Disclaimer: These notes are a draft, and have not been subjected to the usual scrutiny reserved for formal publications

Prophet Inequalities for Matching

Recap Online bipartite matching (OBM)

- Fractional : $1 - 1/e$, tight
- Integral: $1 - 1/e$, tight
- Adwords (small bids): $1 - 1/e$, tight

All adversarial.

Today OBM in a Bayesian environment.

$\boxed{\text{Edge-Arrival}}$

- Bipartite graph $G = (A \sqcup B, E)$ given upfront along with ordering σ of E
- Edge e has weight sampled from $X_e \geq 0$ (known) (all indep.)
- Edges arrive in order $e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(m)}$, reveal weight $w_e \sim X_e$, we must decide irrevocably whether to match

Q Good benchmark?

Today's focus Optimum offline algorithm, OPT_{off}

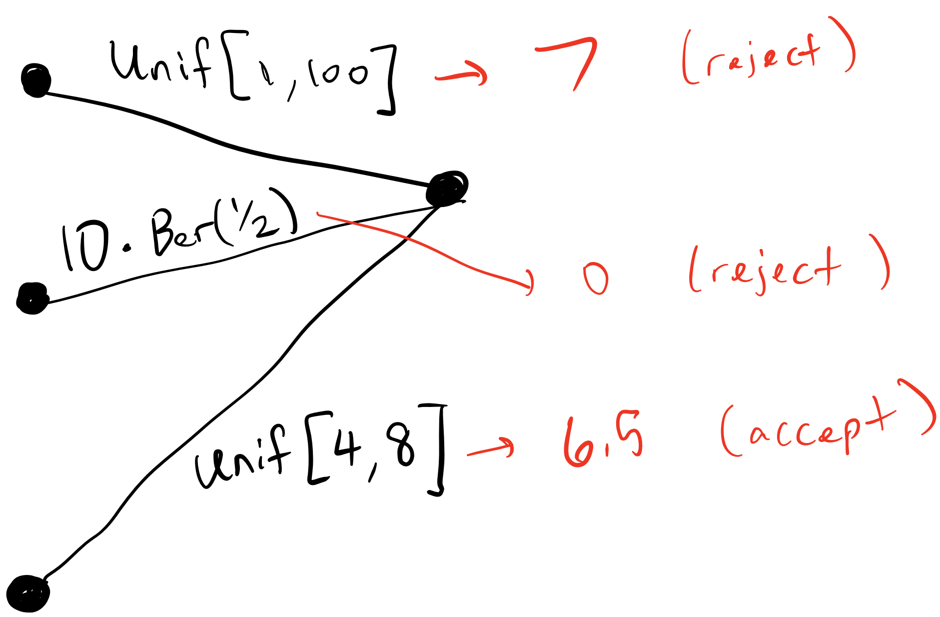
Sees (random) realization of entire graph, gets max-weight matching.

Def Algorithm $\mathcal{A}$ is α-competitive if
$$\mathbb{E}[\mathcal{A}] \geq \alpha \cdot \mathbb{E}[OPT_{off}].$$

OPT_{off} also referred to as <u>prophet</u> (hence the name "prophet inequalities")

Starting Point: Star graph (i.e. single-item)

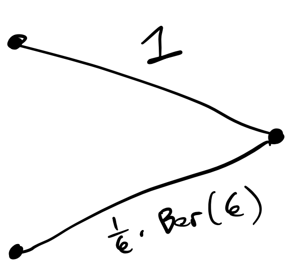
E.g.



<u>Obs</u> Optimal algorithm computable via backwards induction.

<u>Obs</u> No online algo is $> 1/2$ - competitive.

<u>Pf</u>



$$\mathbb{E}[A] = 1$$

$$\mathbb{E}[OPT_{off}] = \epsilon(1/\epsilon) + (1-\epsilon)1$$
$$= 2 - \epsilon$$

Thm [KSG'78] There is always a $1/2$-competitive algorithm.

We will see a proof via Online Contention Resolution Schemes (OCRS).

To motivate this, we introduce the ex-ante relaxation.

$g_e : [0,1] \to \mathbb{R}_{\geq 0}$

$g_e : p \longmapsto \mathbb{E}\left[X_e \,\middle|\, \begin{array}{l} X_e \text{ realized in top } p \\ \text{quantile of its distribution} \end{array}\right]$.

Ex-ante Relaxation

$$\max \sum_e g_e(x_e)$$

$$\text{s.t.} \quad x_e \geq 0, \quad \sum_e x_e \leq 1.$$

Obs. Ex-ante-OPT $\geq \mathbb{E}[OPT_{off}]$.

Strategy: Let x_e^* be optimum. Conditioned on e realizing in top x_e^* quantile, accept w.p. $\geq 1/2$.

[FSZ'16]

Single-item OCRS Given n elts $e_1, \ldots, e_n$ where e_i is active independently w.p. $x_i \geq 0$, and $\sum_{i=1}^{n} x_i \leq 1$.

- Edges arrive in order and reveal their active status
- Want to design an algorithm to accept ≤ 1 active element.

<u>Def</u> $\mathcal{A}$ is <u>c-balanced</u> if for all i,

$$\mathbb{P}[\mathcal{A} \text{ selects } e_i] \geq c \cdot x_i .$$

<u>Claim</u> There exists a ½-balanced single-item OCRS.

<u>Pf</u> Upon arrival of e_i, if no edge selected so far <u>and</u> e_i active, choose it w.p.

$$\frac{1/2}{1 - \frac{1}{2}\sum_{i'<i} x_{i'}} .$$

Show $\mathbb{P}[\text{select } e_i] = \frac{1}{2} x_i$ by induction on i.

<u>Online algo</u>. Let $\{x_e^*\}$ be optimum for ex-ante.
Treat e as "active" iff X_e realizes in top x_e^*-quantile.
Run ½-balanced OCRS

Next Edge-arrival matching via edge-arrival OCRS.

- $\{x_e\}$ now in matching polytope

Natural c-balanced algorithm

If $e=(uv)$ is active and feasible to add, do so w.p. $c/\mathbb{P}[u,v \text{ free}]$.

Claim [EFGT'20]. Above is well-defined with $c=1/3$.

Pf Union bound!

$$\mathbb{P}[u,v \text{ free}] \geq 1-\mathbb{P}[u \text{ matched}]+\mathbb{P}[v \text{ matched}]$$

(by induction) $\longrightarrow$
$$= 1 - c\Big(\sum_{\substack{e'<e \\ e'\in N(u)}} x_{e'} + \sum_{\substack{e'<e \\ e'\in N(v)}} x_{e'}\Big)$$

$$\geq 1-2c$$

So
$$\frac{c}{\mathbb{P}[u,v \text{ free}]} \leq \frac{c}{1-2c} = 1 \quad . \quad \blacksquare$$

for $c=1/3$

Note Current bounds on comp. ratio (bipartite): $[0.341, \frac{3}{7}]$

Prophet Secretary for matching

Motivation: how important is <u>adversarial</u> order?

<u>Starting point</u> Star graph (i.e., single-item)

$\{1, \frac{1}{\epsilon}\cdot Ber(\epsilon)\}$ only gives hardness of $3/4$

Ⓠ Can we beat $1/2$?

Ⓐ Yes!

<u>Single-item RO-OCRS</u> e active w.p. x_e, e's arrive in uniformly random order, want

$\mathbb{P}[\text{select active } e] \geq c \cdot x_e$

<u>Obs</u> $c \leq 1 - 1/e$

<u>Pf</u> Consider $x_e = 1/n$ for n elts e

For that example, taking the first active elt. works. Not always, however...

say $x_1 = \epsilon$, $x_2 = 1-\epsilon$

$\mathbb{P}[\text{take } x_1] = \frac{1}{2}\cdot\epsilon + \frac{1}{2}\cdot\epsilon\cdot\epsilon = \epsilon\left(\frac{1}{2} + \frac{\epsilon}{2}\right).$

Thm [LS'18] Can achieve $c = 1 - 1/e$.

Pf Sample random arrival times $t_1, \ldots, t_n \sim \mathrm{Unif}[0,1]$ independently.

Algo
Upon arrival of e_i, if active take w.p. $\underbrace{e^{-x_i t_i}}_{\text{downsampling}}$

Analysis Analyze

$(*)$ $\mathbb{P}[e_i \text{ available} \mid t_i = y]$

This happens iff for every $j \neq i$,
$t_j > y$, e_j inactive, or e_j did not survive downsampling.

$$(*) = \prod_{j \neq i} \left(1 - \int_0^y x_j e^{-z x_j} \, dz\right)$$

$$= \prod_{j \neq i} \left(1 - \left[e^{-z x_j}\right]_y^0\right)$$

$$= \prod_{j \neq i} e^{-y x_j} = e^{-y \sum_{j \neq i} x_j}$$

$$\mathbb{P}[\text{select } e_i] = \int_0^1 x_i \cdot e^{-x_i y} \cdot e^{-y \sum_{j \neq i} x_j} \, dy$$

$$\geq x_i \int_0^1 e^{-y} \, dy = x_i \left(1 - 1/e\right).$$

Discussion $1-1/e$ tight for RO-OCRS, but still gaps in knowledge for prophet secretary

Esfandiari et al '15: $[1-1/e, \; 0.75]$

Azar et al '18: $\geq 1 - 1/e + 1/400$

Correa et al '21: $[0.669, \; 0.732]$

Beyond star-graphs

Claim Can achieve a 0.432-approx. for matching in general graphs.

Pf sketch Apply same downsampling as for single-item!